

# and Probabilistic Computations

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## Abstract

The main purpose of this paper is to show that we can exploit the difference ( $l_1$ -norm and  $l_2$ -norm) in the probability calculation between quantum and probabilistic computations to claim the difference in their space efficiencies. It is shown that there is a finite language  $L$  which contains sentences of length up to  $O(n^{c+1})$  such that: (i) There is a one-way quantum finite automaton (qfa) of  $O(n^{c+4})$  states which recognizes  $L$ . (ii) However, if we try to simulate this qfa by a probabilistic finite automaton (pfa) *using the same algorithm*, then it needs  $\Omega(n^{2c+4})$  states. It should be noted that we do not prove real lower bounds for pfa's but show that if pfa's and qfa's use exactly the same algorithm, then qfa's need much less states.

## 1 Introduction

It is a fundamental rule of quantum computation that if a state  $q$  has an amplitude of  $\sigma$ , then  $q$  will be observed not with probability  $\|\sigma\|$  but with probability  $\|\sigma\|^2$ . Therefore, if one can increase the amplitude of  $q$  twice, i.e., from  $\sigma$  to  $2\sigma$ , then the corresponding probability increases four times, i.e.,  $\|\sigma\|^2$  to  $4\|\sigma\|^2$ . In general, if the amplitude increases  $k$  times then the probability increases  $k^2$  times. One observation of the Grover search [Gro96], which is apparently one of the most celebrated quantum algorithms, is that it takes advantage of this fact cleverly, by inventing the (efficient) quantum process whose  $k$  iterations increase the amplitude of a designated state roughly  $k$  times. As described above, this is equivalent to increasing the probability  $k^2$  times. Thus the Grover search is faster quadratically than the classic randomized search whose  $k$  iterations can increase the probability only  $k$  times.

In this paper, we also exploit this feature, i.e., the difference in probability calculation between quantum and probabilistic computations, but from a bit different angle: Suppose that there are ten pairs of state  $(p_1, q_1), \dots, (p_{10}, q_{10})$  where, for each  $1 \leq i \leq 10$ , either  $p_i$  or  $q_i$  has the amplitude  $1/\sqrt{10}$  (we say that  $p_i$  is ON if it has the amplitude and OFF otherwise.). We wish to know how many  $p_i$ 's are ON. This can be done by "gathering" amplitudes by applying a Fourier transform from  $p_i$ 's to  $r_i$ 's and by observing  $r_{10}$  (see later sections for details). If all ten  $p_i$ 's are ON, then the amplitude of  $r_{10}$  after Fourier transform is one and it is observed with probability one. If, for example, only three  $p_i$ 's are ON, then the amplitudes of  $r_{10}$  is  $3/10$  and is observed with probability  $9/100$ . In the case of probabilistic computation, we can also gather the probability of  $p_i$ 's ( $= 1/10$  for each) simply by defining (deterministic) transitions from  $p_i$  to  $r_{10}$ . If all pairs are ON, then the probability that  $r_{10}$  is observed is one again, but if only three  $p_i$ 's are ON, its probability is  $3/10$ . If the latter case (only three  $p_i$ 's are ON) is associated with some erroneous situation, this probability,  $3/10$ , is much larger than  $9/100$  in the quantum case. In other words quantum computation can enjoy much smaller error-probability due to the difference in the rule of probability calculation.

The question is of course whether we can turn this feature into some concrete result or how we can translate this difference in probability into some difference in efficiency like time and space. In this paper we give an affirmative answer to this question by using quantum finite automata; we prove that there is a finite language  $L$  which contains sentences of length up to  $O(n^{c+1})$  such that: (i) There is a one-way quantum finite automaton (qfa) of  $O(n^{c+4})$  states which recognizes  $L$ . (ii) However, if we try to simulate this qfa by a probabilistic finite automaton (pfa) *using the same algorithm*, then it needs  $\Omega(n^{2c+4})$  states. It should be noted that we do not prove real lower bounds for pfa's but show that if pfa's and qfa's use exactly the same algorithm (the only difference is the way of gathering amplitudes mentioned above), then qfa's need much less states.

Quantum finite automata have been popular in the literature since its simplicity is nice to understand merits and demerits of quantum computation[AF98, AG00, AI99, ANTV99, KW97, Nay99]. Ambainis and

Freivalds [AF98] proved an exponential difference in the size of qfa's and pfa's for a one-letter language. Their technique highly depends on the *rotation of complex amplitudes*, which is exceptionally powerful for a certain situation. Nayak[Nay99] gave a negative side of qfa's by showing  $L_n = \{wa \mid w \in \{a, b\}^* \text{ and } |w| \leq n\}$  needs exponentially more states for qfa's than for dfa's.  $L_\infty$  is a regular set but is not recognizable by qfa's as shown by Kondacs and Watrous[KW97]. [KW97] also introduced 2-way qfa's which can recognize non-regular languages. To our best knowledge, the notion of gathering amplitudes using Fourier transform appeared in this paper for the first time and played an important role in [AI99], too.

## 2 Problem EQ

Suppose that Alice and Bob have  $n$ -bit numbers  $x$  and  $y$  and they wish to know whether or not  $x = y$ . This problem, called EQ, is one of the most famous problems for which its randomized communication complexity ( $= \Theta(\log n)$ ) is significantly smaller than its deterministic counterpart ( $= n + 1$ ) [KN97]. In this paper, we need a little bit more accurate argument on the value of randomized (and one-way) communication complexity: Consider the following protocol  $M_{EQ}$ : (i) Alice selects a single prime  $p$  among the smallest  $N$  primes. (ii) Then she divides  $x$  by  $p$  and sends Bob  $p$  and the residue  $a$ . (iii) Bob also divides his number  $y$  by  $p$  and compares his residue with  $a$ . They accept  $(x, y)$  iff those residues coincide.

It is obvious that if  $x = y$  then protocol  $M_{EQ}$  accepts  $(x, y)$  with probability one. Let  $E(N)$  be the maximum (error) probability that  $M_{EQ}$  accepts  $(x, y)$  even if  $x \neq y$ . To compute  $E(N)$ , we need the following lemma: (In this paper,  $\log n$  always means  $\log_2 n$  and  $\lceil f(n) \rceil$  for a function  $f(n)$  is simply written as  $f(n)$ .)

**Lemma 1.** Suppose that  $x \neq y$  and let  $S(x, y)$  be a set of primes such that  $x = y \pmod p$  for all  $p$  in  $S(x, y)$ . Also, let  $s$  be the maximum size of such a set  $S(x, y)$  for a pair of  $n$ -bit integers  $x$  and  $y$ . Then  $s = \Theta(n/\log n)$ .

**Proof.** Let  $p_i$  be the  $i$ -th largest prime and  $\pi(n)$  be the number of different primes  $\leq n$ . Then the prime number theorem says that  $\lim_{n \rightarrow \infty} \frac{\pi(n)}{n/\log_e n} = 1$ , which means that  $p_{n/\log n} = \Theta(n)$ . Consequently, there must be a constant  $c$  s.t.  $p_1 \cdot p_2 \cdots p_{n/\log n} \cdots p_{cn/\log n} > p_{n/\log n} \cdot p_{n/\log n+1} \cdots p_{cn/\log n} > 2^n$  since  $n^{n/\log n} = 2^n$ . Thus an  $n$ -bit integer  $z$  has at most  $cn/\log n$  different prime factors. Note that  $x = y \pmod a$  iff  $|x - y| = 0 \pmod a$ . Hence,  $s \leq cn/\log n$ . Also it turns out by the prime number theorem that there is an  $n$ -bit integer  $z$  such that it has  $c'n/\log n$  different prime factors for some constant  $c'$ , which proves that  $s \geq c'n/\log n$ . ■

In this paper,  $N_0$  denotes this number  $s$  which is  $\Theta(n/\log n)$ . Then

**Lemma 2.**  $E(N) = N_0/N$ .

For example, if we use  $N = n^2/\log n$  different primes in  $M_{EQ}$ , its error-rate is  $1/n$ .

## 3 Our Languages and qfa's

A one-way qfa is the following model: (i) Its input head always moves one position to the right each step. (ii) Global state transitions must be unitary. (iii) Its states are partitioned into *accepting*, *rejecting* and *non-halting* states. (iv) Observation is carried out every step, and if acceptance or rejection is observed, then the computation halts. Otherwise, computation continues after proportionally distributing the amplitudes of accepting and rejecting states to non-halting states. We omit the details, see for example [KW97]. In this paper, we consider the following three finite languages.

$$L_0(n) = \{w\#w^R \mid w \in \{0, 1\}^n\},$$

$$L_1(n) = \{w_1\#w_2\#\#w_3\#w_4\# \mid w_1, w_2, w_3, w_4 \in \{0, 1\}^n, (w_1 = w_2^R) \vee ((w_1w_2) = (w_3w_4)^R)\},$$

$$L_2(n, k) = \{w_{11}\#w_{12}\#\#w_{13}\#w_{14}\#\#\# \cdots \#\#\#w_{i1}\#w_{i2}\#\#w_{i3}\#w_{i4}\#\#\# \cdots \#\#\#w_{k1}\#w_{k2}\#\#w_{k3}\#w_{k4}\# \mid$$

$$w_{i1}, w_{i2}, w_{i3}, w_{i4} \in \{0, 1\}^n, 1 \leq i \leq k$$

$$\text{and } 1 \leq \exists j \leq k \text{ s.t. } (w_{j1} = w_{j2}^R) \wedge (\text{for all } 1 \leq i \leq j-1, (w_{i1}w_{i2}) = (w_{i3}w_{i4})^R)\}.$$

In the next section, we first construct a qfa  $M_0^Q$ , which accepts each string  $x \in L_0$  with probability 1 and each string  $y \notin L_0$  with probability at most  $\frac{1}{n}$ .  $M_0^Q$  simulates the protocol  $M_{EQ}$  in the following way (see Fig 1). Given an input string  $\phi w_1\#w_2\$$  ( $\phi$  is the leftmost and  $\$$  is the rightmost symbols),  $M_0^Q$  first splits

into  $N$  different states  $q_{p_1}, \dots, q_{p_i}, \dots, q_{p_N}$  with equal amplitudes by reading  $\phi$ . Then from  $q_{p_i}$ , submachine  $M_{1i}$  starts the task for dividing integer  $w_1$  by the  $i$ -th prime  $p_i$ . This computation ends up in some state of  $M_{1i}$  which corresponds to the residue of the division. This residue is shifted to the next submachine  $M_{2i}$ , and then  $M_{2i}$  carries out a completely opposite operation while reading  $w_2$ . If (and only if) two residues are the same,  $M_{2i}$  ends up in some specific state  $q_i^0$ .  $M_0^Q$  then applies a Fourier transform from  $q_i^0$  to  $s_i$  for  $1 \leq i \leq N$ .  $M_0^Q$  thus simulates  $M_{EQ}$  by setting  $s_N$  as its only accepting state.

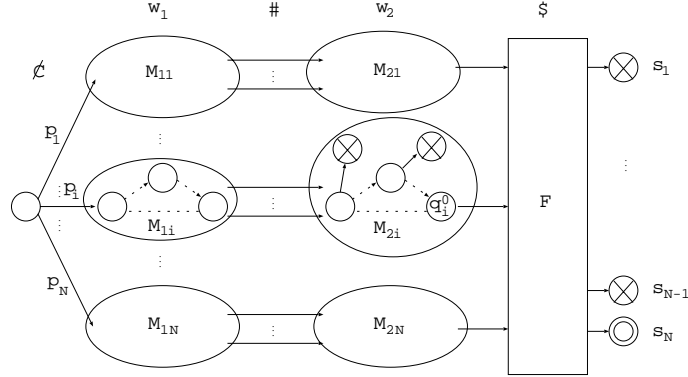


Fig 1. qfa  $M_0^Q$

For the probabilistic counterpart, pfa  $M_0^P$ , we can use exactly the same state transition, except for deterministic transitions from  $q_i^0$  to  $s_N$ . As mentioned before we can achieve a quadratic difference in the probability of error, like  $(1/n)^2$  for  $M_0^Q$  v.s.  $(1/n)$  for  $M_0^P$ . It would be nice if this quadratic difference of error can be traded directly to a quadratic difference in the necessary number of primes or to a quadratic difference in the size of automata. Unfortunately that is not possible: The main reason is that we do not need such a small (like  $1/n$  or  $1/n^2$ ) error-rate but something like  $1/3$  is enough by definition. Then the quadratic difference in the error is reduced to a difference between, say,  $1/3$  and  $1/9$ , which means only a difference of the constant factor in the necessary number of primes or the necessary number of states.

There is a standard technique to overcome this difficulty, namely, the use of iteration. Consider the following string:

$$w_{11}\#w_{12}\#\#w_{21}\#w_{22}\#\#\dots\#w_{n1}\#w_{n2}$$

where the accepting condition is that for some  $1 \leq j \leq n$ ,  $w_{j1} = w_{j2}^R$ . When all pairs  $(w_{j1}, w_{j2})$  do not satisfy this condition, the (error) probability of accepting such a string is roughly  $O(\frac{1}{n}) \times n = O(1)$ , which appears desirable for our purpose.

This argument does not seem to cause any difficulty for pfa's but it does for qfa's for the following reason: After checking  $w_{11}$  and  $w_{12}$ , the qfa is in a single accepting state if the condition is met, which is completely fine. However, if  $w_{11} \neq w_{12}^R$  and the observation is not accepting, then there are many small amplitudes distributed to many different states. Note that we have to continue the calculation for  $w_{21}$  and  $w_{22}$  which should be started from a single state. (It may be possible to start the new computation from each non-halting state, but that will result in an exponential blow-up in the number of states, which again implies no clear separation in the size of automata.) One can see easily that we cannot use a Fourier transform this time to gather the amplitudes since there are many different patterns in the distribution of states which have a small nonzero amplitudes.

This is the reason why the next language  $L_1(n)$  plays an important role. Suppose that  $w_1 \neq w_2^R$ . Then the resulting distribution of amplitudes is quite complicated as mentioned above. However, no matter how it is complicated, we can completely "reverse" the previous computation for  $w_1\#w_2$  by reading  $w_3\#w_4$  if  $(w_1w_2) = (w_3w_4)^R$ . This reverse computation should end up in a single state of amplitude one (actually it is a little less than one) since the original computation for  $w_1 \neq w_2^R$  starts from the (single) initial state. Now one can use the iteration scheme, which naturally leads us to the third language  $L_2(n, k)$ .

## 4 Main Results

As mentioned in the previous section, we construct our qfa's and corresponding pfa's for  $L_0(n)$ ,  $L_1(n)$  and  $L_2(n, n^c)$  in a step-by-step fashion. Recall that  $N$  is the number of primes used in protocol  $M_{EQ}$  and  $N_0 = \Theta(n/\log n)$ .

**Lemma 3.** There exists a qfa  $M_0^Q$  which accepts strings in  $L_0$  with probability one and strings not in  $L_0$  with probability at most  $(\frac{N_0}{N})^2$ . The number of states in  $M_0^Q$  is  $\Theta(N^2 \log N)$ .

**Proof.**  $M_0^Q$  has the following states: (i) An initial state  $q_0$ , (ii)  $q_{p_k, j_k, 1}$  (in submachine  $M_{1i}$  of Fig 1), (iii)  $q_{p_k, j_k, 2}$  (in  $M_{2i}$  of Fig 1), (iv)  $q_{p_k, j_k, rej}$  (also in  $M_{2i}$  of Fig 1), (v)  $s_l$ , where  $1 \leq k \leq N$ ,  $0 \leq j_k \leq p_k - 1$  and  $1 \leq l \leq N$ .  $p_k$  denotes the  $k$ -th largest prime  $> 3$  (two is excluded for the reason mentioned later).  $s_N$  is the only accepting state,  $q_{p_k, j_k, rej}$  and  $s_l$  ( $1 \leq l \leq N - 1$ ) are rejecting states and all the others are non-halting states. We give a complete state transition diagram of  $M_0^Q$  in Table 1, where  $V_\sigma|Q\rangle = \alpha_1|Q_1\rangle + \dots + \alpha_i|Q_i\rangle + \dots + \alpha_m|Q_m\rangle$  means that if  $M_0^Q$  reads symbol  $\sigma$  in state  $Q$ , it moves to each state  $Q_i$  with amplitude  $\alpha_i$  ( $|\alpha_1|^2 + \dots + |\alpha_m|^2 = 1$ ).

When reading  $\phi$  of the input string  $\phi w_1 \# w_2 \$$ ,  $M_0^Q$  splits into  $N$  submachines (denoted by  $M_{1i}$  in Fig 1) with equal amplitudes (see transition (1) of Table 1). The  $k$ -th submachine  $M_{1k}$  computes the residue when dividing  $w_1$  by  $p_k$  (by using transition (2 - a) to (2 - d) in Table 1). This division can be done simply by simulating the usual division procedure as shown in Fig 2 (a) and (b) for  $w_1 = 110001$  and  $p_2 = 101$  ( $= 5$ ). State  $j$  in Fig 2 (b) corresponds to  $q_{p_2, j, 1}$ . The starting state is 0 and by reading the first symbol 1 it goes to state 1. By reading the second symbol 1, it goes to state 3 ( $= 11$ ). Now reading 0, it goes to state 1 since  $110 = 1 \pmod{101}$ . This continues until reading the last symbol 1 and  $M_0^Q$  ends up in state 4. It should be noted that these state transitions are reversible: For example, if the machine reaches state 2 ( $= 10$ ) from some state  $Q$  by reading 0, then  $Q$  must be state 1 since  $Q$  cannot be greater than 2. (Reason: If  $Q$  is greater than 2, it means that the quotient will be 1 after reading a new symbol. Since  $M_0^Q$  reads 0 as the new symbol, the least significant bit of the residue when divided by 5 must be 1, which excludes state 2 as its next state.) Hence the quotient must have been 0, and so the previous state must be 1. Note that this argument holds because we excluded two, which is the only even prime, from  $p_k$ .

Thus, if  $w_1 \pmod{p_k} = j_k$ , then  $M_0^Q$  is in superposition  $\frac{1}{\sqrt{N}} \sum_{k=1}^N |q_{p_k, j_k, 1}\rangle$  after reading  $w_1$ . Then  $M_0^Q$  reads  $\#$  and this superposition is "shifted" to  $\frac{1}{\sqrt{N}} \sum_{k=1}^N |q_{p_k, j_k, 2}\rangle$ , where  $M_0^Q$  checks if  $w_2^R \pmod{p_k}$  is also  $j_k$  by using transition (4 - a) to (4 - d) in Table 1. This job can be done by completely reversing the previous procedure of dividing  $w_1$  by  $p_k$ . Actually, the state transitions are obtained by simply reversing the directions of previous state diagrams. Since previous transitions are reversible, new transitions are also reversible. Now one can see that the  $k$ -th submachine  $M_{2k}$  is in state  $q_{p_k, 0, 2}$  iff the two residues are the same.

Finally by reading  $\$$ , Fourier transform is carried out only from these zero-residue states  $q_{p_k, 0, 2}$  to  $s_l$ . From other states  $q_{p_k, j, 2}$  ( $j \neq 0$ )  $M_0^Q$  goes to rejecting states  $q_{p_k, j, rej}$ . If the residues are the same in only  $t$  submachines out of the  $k$  ones, the amplitude of  $s_N$  is computed as

$$\frac{1}{N} \sum_{|t|} \sum_{l=1}^N \exp\left(\frac{2\pi i}{N} kl\right) |s_l\rangle = \frac{t}{N} |s_N\rangle + \frac{1}{N} \sum_{|t|} \sum_{l=1}^{N-1} \exp\left(\frac{2\pi i}{N} kl\right) |s_l\rangle,$$

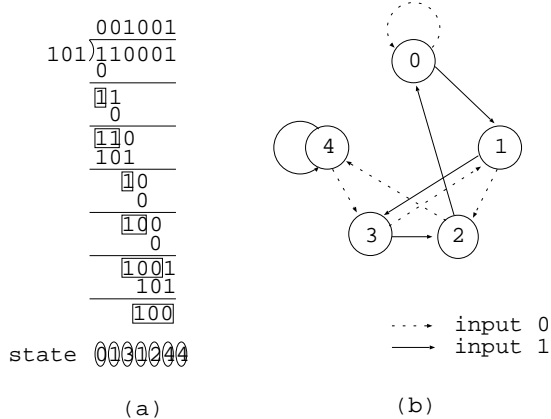
namely that is equal to  $t/N$ . Thus the probability of acceptance is  $(\frac{t}{N})^2$ . If the input string is in  $L_0$ , then this probability becomes 1. Otherwise, it is at most  $(N_0/N)^2$  by Lemma 2. The number of states in  $M_0^Q$  is given as

$$1 + 2 \sum_{k=1}^N p_k + \sum_{k=1}^N (p_k - 1) + N = 1 + 3 \sum_{k=1}^N p_k \leq 1 + 3 \cdot N \cdot p_N = O(N^2 \log N),$$

which completes the proof. ■

$$\begin{aligned}
(1) \quad & V_\ell |q_0\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^N |q_{p_k,0,1}\rangle, & (4-a) \quad & V_0 |q_{p_k,j,2}\rangle = |q_{p_k,\frac{j}{2},2}\rangle \quad (j : \text{even}), \\
(2-a) \quad & V_0 |q_{p_k,j,1}\rangle = |q_{p_k,2j,1}\rangle \quad (0 \leq j < \frac{p_k}{2}), & (4-b) \quad & V_0 |q_{p_k,j,2}\rangle = |q_{p_k,\frac{j+p_k}{2},2}\rangle \quad (j : \text{odd}), \\
(2-b) \quad & V_0 |q_{p_k,j,1}\rangle = |q_{p_k,2j-p_k,1}\rangle \quad (\frac{p_k}{2} < j < p_k), & (4-c) \quad & V_1 |q_{p_k,j,2}\rangle = |q_{p_k,\frac{j-1+p_k}{2},2}\rangle \quad (j : \text{even}), \\
(2-c) \quad & V_1 |q_{p_k,j,1}\rangle = |q_{p_k,2j+1,1}\rangle \quad (0 \leq j < \frac{p_k}{2} - 1), & (4-d) \quad & V_1 |q_{p_k,j,2}\rangle = |q_{p_k,\frac{j-1}{2},2}\rangle \quad (j : \text{odd}), \\
(2-d) \quad & V_1 |q_{p_k,j,1}\rangle = |q_{p_k,2j+1-p_k,1}\rangle \quad (\frac{p_k}{2} - 1 < j < p_k), & (5-a) \quad & V_{\$} |q_{p_k,0,2}\rangle = \frac{1}{\sqrt{N}} \sum_{l=1}^N \exp\left(\frac{2\pi i}{N} kl\right) |sl\rangle, \\
(3) \quad & V_{\sharp} |q_{p_k,j,1}\rangle = |q_{p_k,j,2}\rangle, & (5-b) \quad & V_{\$} |q_{p_k,j,2}\rangle = |q_{p_k,j,rej}\rangle \quad (1 \leq j < p_k).
\end{aligned}$$

**Table 1.** State transition diagram of  $M_0^Q$



**Fig 2. division procedure for  $w_1 = 110001$  and  $p_k = 5$**

Let us consider the pfa whose state transition is exactly the same as  $M_0^Q$  of  $f(N)$  states excepting that the state transitions from  $q_{p_k,0,2}$  to  $s_l$  for the Fourier transform are replaced by simple (deterministic) transitions from  $q_{p_k,0,2}$  to  $s_N$ . We call such a pfa *emulates* the qfa. Suppose that  $M^P$  emulates  $M^Q$ . Then the size of  $M^P$  is almost the same as that of  $M^Q$ , i.e., it is also  $\Theta(f(N))$  if the latter is  $f(N)$ , since the Fourier transform does not make much difference in the number of states. The following lemma is easy:

**Lemma 4.** Suppose that  $M_0^P$  emulates  $M_0^Q$ . Then  $M_0^P$  accepts strings in  $L_0$  with probability one and those not in  $L_0$  with probability  $N_0/N$ . ■

Let us set, for example,  $N = N_0\sqrt{n}$ . Then the error-rate of  $M_0^Q$  is  $(N_0/N)^2 = \frac{1}{n}$  and its size is  $O(n^3/\log n)$ . To achieve the same error-rate by a pfa, we have to set  $N = N_0n$ , which needs  $O(n^4/\log n)$  states.

**Remark.** Suppose that we have once designed a specific qfa  $M_0^Q$  (similarly for  $M_0^P$ ). Then it can work for inputs of any length or it does not reject the input only by the fact that its length is not  $2n + 1$ . The above calculation of the acceptance and rejection rates is only true when our input is restricted to strings  $\subseteq \{0, 1\}^n \# \{0, 1\}^n$ .

The following lemmas, Lemma 5 and 6 (see Acknowledgment), are important for the analysis of error probability of  $M_1^Q$ , a qfa which recognizes the second language  $L_1(n)$ . Here,  $\|\psi\|$  means the norm of a vector  $\psi$  and  $\|\psi\|_{acc}$  the norm of  $\psi$  after being projected onto accepting space, i.e., the accepting probability is  $\|\psi\|_{acc}^2$ .  $\langle \psi | \phi \rangle$  denotes the inner product between  $\psi$  and  $\phi$ .

**Lemma 5.** Let  $\psi$  be a quantum state such that applying a unitary transformation  $U$  followed by a measurement to  $\psi$  causes acceptance with probability 1, i.e.,  $\|U\psi\|_{acc}^2 = 1$ . If  $\psi$  can be decomposed into two orthogonal states  $\psi_1$  and  $\psi_2$  s.t.  $\psi = \psi_1 + \psi_2$ , then  $\|U\psi_1\|_{acc}^2 \geq \|\psi_1\|^4$ .

**Proof.** Let  $H = \text{span}\{\phi \mid \|U\phi\|_{\text{acc}} = 1 \text{ and } \|\phi\| = 1\}$ , i.e.,  $H$  is obtained by applying  $U^{-1}$  to the subspace spanned by accepting states only. The accepting probability of  $U\psi_1$  is equal to the squared projection of  $U\psi_1$  on the subspace spanned by accepting states. Since  $U$  is unitary and any unitary transformation preserves the inner product, it turns out that this projection is the same as the projection of  $\psi_1$  onto  $H$ .

Let  $H' = \text{span}\{\psi\}$ . Since  $\|U\psi\|_{acc}^2 = 1$ , we have  $H' \subseteq H$ . Therefore the projection of  $\psi_1$  to  $H$  is at least the projection of  $\psi_1$  to  $H'$ , namely at least  $\|\langle\psi_1|\psi\rangle\| = \|\psi_1\|^2$ . To summarize,  $\|U\psi_1\|_{acc}^2 \geq \|\langle\psi_1|\psi\rangle\|^2 = \|\psi_1\|^4$ . ■

**Lemma 6.** Let  $\psi$  be a quantum state such that applying a unitary transformation  $U$  followed by a measurement to  $\psi$  causes acceptance with probability at most  $\alpha^2$ , i.e.,  $\|U\psi\|_{acc}^2 \leq \alpha^2$ . If  $\psi$  can be decomposed into two orthogonal states  $\psi_1$  and  $\psi_2$  s.t.  $\psi = \psi_1 + \psi_2$ , then

$$\|U\psi_1\|_{acc}^2 \leq \|\psi_1\|^2 (\alpha\|\psi_1\| + \|\psi_2\|)^2.$$

**Proof.** Let  $H$  be the Hilbert space spanned by  $\psi_1$  and  $\psi_2$ . Then,  $\psi_1$  can also be written as

$$\psi_1 = \langle\psi|\psi_1\rangle \psi + \langle\bar{\psi}|\psi_1\rangle \bar{\psi},$$

where  $\bar{\psi}$  is a normalized vector in  $H$  and perpendicular to  $\psi$ . Note that  $\|\psi\|$  is also 1. Again  $\|\langle\psi|\psi_1\rangle\| = \|\psi_1\|^2$  and from the above equation we obtain that  $\langle\psi|\psi_1\rangle = \|\langle\psi|\psi_1\rangle\|^2 + \|\langle\bar{\psi}|\psi_1\rangle\|^2$ , which implies that  $\|\langle\bar{\psi}|\psi_1\rangle\|^2 = \|\psi_1\|^2(1 - \|\psi_1\|^2)$ . Note that  $\|\psi_1\|^2 + \|\psi_2\|^2 = 1$ . Thus,  $\|\langle\bar{\psi}|\psi_1\rangle\| = \|\psi_1\|\|\psi_2\|$ . Since  $U\psi_1 = \langle\psi|\psi_1\rangle U\psi + \langle\bar{\psi}|\psi_1\rangle U\bar{\psi}$  and our observation is a simple projection, it follows by triangular inequality that

$$\begin{aligned} \|U\psi_1\|_{acc} &\leq \|\langle\psi|\psi_1\rangle\| \|U\psi\|_{acc} + \|\langle\bar{\psi}|\psi_1\rangle\| \|U\bar{\psi}\|_{acc} \\ &\leq \|\psi_1\|^2 \alpha + \|\psi_1\| \|\psi_2\| \\ &= \|\psi_1\| (\alpha\|\psi_1\| + \|\psi_2\|). \end{aligned}$$

This proves the lemma. ■

Now we shall design a qfa  $M_1^Q$  which recognizes the second language  $L_1(n)$ .

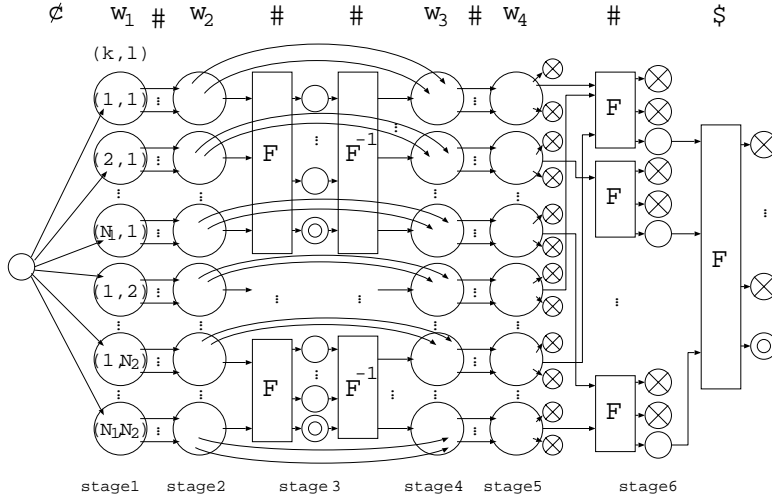
**Lemma 7.** There exists a qfa  $M_1^Q$  which accepts strings in  $L_1$  with probability at least  $1 - (\frac{N_0}{N_1})^2 + (\frac{N_0}{N_1})^4$  and strings not in  $L_1$  with at most  $(\frac{N_0}{N_1})^2 + (1 - (\frac{N_0}{N_1})^2)(\frac{N'_0}{N_2} + \frac{N_0}{N_1})^2$ .  $M_1^Q$  has  $\Theta((N_1 N_2)^2 \log N_1 \cdot \log N_2)$  states. Here  $N'_0$  denotes the number  $s$  in Lemma 1 but for  $x$  and  $y$  of length  $2n$ .

**Proof.** Again a complete state transition diagram is shown in Table 2, where accepting states are  $s_{N_1,0,p_l,f}$  such that  $0 \leq f \leq p_l - 1$  and  $t_{N_1}$ . Rejecting states are  $q_{p_k,e,p_l,f,rej}$  such that  $e \neq 0$  or  $f \neq 0$ ,  $0 \leq e \leq p_k - 1$ ,  $0 \leq f \leq p_l - 1$ ,  $t_{p_k,0,y}$  such that  $1 \leq y \leq N_2 - 1$ , and  $t_z$  such that  $1 \leq z \leq N_1 - 1$ . All other states are non-halting.

$M_1^Q$  checks whether  $w_1 = w_2^R$  using  $N_1$  primes and also whether  $(w_1 w_2) = (w_3 w_4)^R$  using  $N_2$  primes. Note that those two jobs have to be done at the same time using composite automata while reading  $w_1 \# w_2$ . Hence  $M_1^Q$  first splits into  $N_1 \cdot N_2$  submachines, each of which is denoted by  $M(k, l)$ ,  $1 \leq k \leq N_1$ ,  $1 \leq l \leq N_2$ . As shown in Fig 3,  $M(k, l)$  has six stages, from stage 1 thorough stage 6. It might be convenient to think that each state of  $M(k, l)$  be a pair of state  $(q_L, q_R)$  and to think  $M(k, l)$  be a composite of  $M_L$  and  $M_R$ . In stages 1 and 2,  $M_L$  has similar state transitions to those of Table 1 for checking  $w_1 \neq w_2^R$ .  $M_R$  has also similar transitions but only for the first part of it, i.e., to compute  $w_1 w_2 \bmod p_l$ . This portion of transitions are given in (2) to (4) of Table 2.

Now we go to stage 3. Here  $M_L$ , reading the first  $\#$ , carries out the Fourier transform exactly as  $M_0^Q$  (see (5-a) in Table 2). After that  $M_L$ , reading the second  $\#$ , execute Inverse Fourier transform from states  $s_{m,0,p_l,f}$  ( $1 \leq m \leq N_1$ ), which is shown in (6-a) of Table 2. In this stage,  $M_R$  does nothing; it just shifts the state information about  $(w_1 w_2) \bmod p_l$  (but only when  $w_1 \neq w_2^R$ ) to stage 4.

Stages 4 and 5 are for the complete reverse operation of stages 2 and 1, respectively. By doing this, the amplitudes for state  $q_L$ , which were once in turmoil after stage 2, are reorganized and gathered in specific states, namely  $q_{p_k,0,p_l,0,4}$  if  $(w_1 w_2) = (w_3 w_4)^R$ . Therefore, what we do is to gather the amplitude of  $q_{p_k,0,p_l,0,4}$  to  $t_{p_k,0,N_2}$  by Fourier transform reading  $\#$ . Now reading the rightmost symbol, we do another Fourier transform, which gathers the amplitudes of  $t_{p_k,0,N_2}$  to  $t_{N_1}$ .



**Fig 3.** qfa  $M_1^Q$

For the analysis of error probability, Lemma 5 and 6 are convenient. The basic idea is as follows: When  $w_1 \neq w_2^R$ , a small amplitude,  $\frac{1}{\sqrt{N_2}} \frac{N_0}{N_1}$  is “taken” by each of the  $N_2$  accepting states in stage 3. This is basically the same as  $M_0^Q$  since its probability of observing acceptance is  $\sum_{l=1}^{N_2} \left( \frac{1}{\sqrt{N_2}} \cdot \frac{N_0}{N_1} \right)^2 = \left( \frac{N_0}{N_1} \right)^2$ . So, the problem is how much of the remaining amplitudes distributed on the other states in this stage can be retrieved in the final accepting state  $t_{N_1}$  when  $(w_1 w_2) = (w_3 w_4)^R$ .

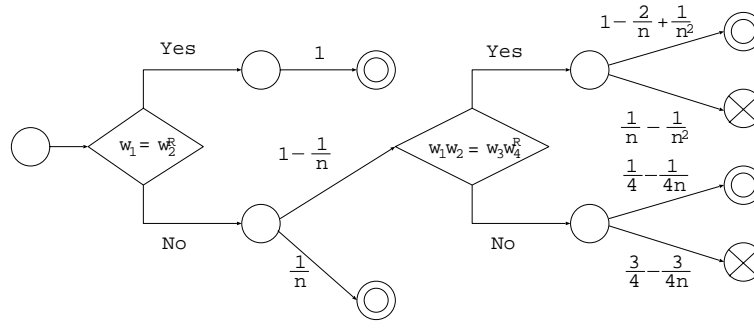
Suppose that we construct a new qfa  $M'$  which is exactly the same as  $M_1^Q$  but all the  $N_2$  halting states of  $M_1^Q$  in stage 3 are changed to non-halting states. Thus  $M'$  only checks the longer strings, whether  $(w_1 w_2) = (w_3 w_4)^R$  or not. It is clear that  $M'$  accepts with probability exactly one when  $(w_1 w_2) = (w_3 w_4)^R$  and with probability at most  $(\frac{N'_0}{N_2})^2$  when  $(w_1 w_2) \neq (w_3 w_4)^R$ .

Note that Lemma 5 and 6 also hold for any sequence of unitary transformations and measurements since we can delay measurements and replace them with a single unitary transformation  $U$  followed by a measurement[ANTV99]. Next, consider (i) $\psi$ , (ii) $\psi_1$  and (iii) $\psi_2$  in Lemma 5 and 6 as (i)the quantum state, (ii)the superposition of non-halting states and (iii)the superposition of accepting states in stage 3 of  $M_1^Q$  right after Fourier transform, respectively. In our case,  $\|\psi_1\|^2 = (1 - (\frac{N_0}{N_1})^2)$ ,  $\|\psi_2\|^2 = (\frac{N_0}{N_1})^2$  and  $\alpha^2 = (\frac{N'_0}{N_2})^2$ . Thus, from Lemma 5 we can obtain that when  $(w_1 w_2) = (w_3 w_4)^R$ ,  $M_1^Q$  accepts with probability at least  $\left( \frac{N_0}{N_1} \right)^2 + \left( 1 - \left( \frac{N_0}{N_1} \right)^2 \right)^2$ . Also from Lemma 6, when  $(w_1 w_2) \neq (w_3 w_4)^R$ ,  $M_1^Q$  accepts with probability at most  $\left( \frac{N_0}{N_1} \right)^2 + \left( 1 - \left( \frac{N_0}{N_1} \right)^2 \right) \left( \sqrt{\left( 1 - \left( \frac{N_0}{N_1} \right)^2 \right) \frac{N'_0}{N_2} + \frac{N_0}{N_1}} \right)^2 \leq \left( \frac{N_0}{N_1} \right)^2 + \left( 1 - \left( \frac{N_0}{N_1} \right)^2 \right) \left( \frac{N'_0}{N_2} + \frac{N_0}{N_1} \right)^2$ .

Finally, we count the number of states in  $M_1^Q$ . This is not hard since the whole machine is a composition of two machines, one for using  $N_1$  different primes and the other for  $N_2$  different primes. Therefore the size of the composite machine is  $O((N_1 N_2)^2 \log N_1 \cdot \log N_2)$ . ■

Suppose that we set  $N_1 = N_0 \sqrt{n}$ , and  $N_2 = d N'_0$ . Then  $\frac{N_0}{N_1} = \frac{1}{\sqrt{n}}$  and  $\frac{N'_0}{N_2} + \frac{N_0}{N_1} \leq \frac{1}{2}$  if we select a sufficiently large constant  $d$ , e.g.  $d = 3$  when  $n \geq 36$ . Namely,  $M_1^Q$  accepts strings in  $L_1$  with probability at least  $1 - 1/n + 1/n^2$  and those not in  $L_1$  with probability at most  $\frac{1}{4} + \frac{3}{4n}$ . The number of states is  $\Theta(\frac{n^5}{\log^2 n})$ . The probability distribution for each state of  $M_1^Q$  is illustrated in Fig 4 (for example, above  $1 - 1/n + 1/n^2$  is the sum of  $1/n$  and  $1 - 2/n + 1/n^2$ ).

Let us consider pfa  $M_1^P$  which recognizes  $L_1(n)$ . The state transition of  $M_1^P$  is the same as that of  $M_1^Q$  except for Fourier transform and Inverse Fourier transform only  $M_1^Q$  performs. If string  $x$  satisfies  $w_1 \neq w_2^R$ ,

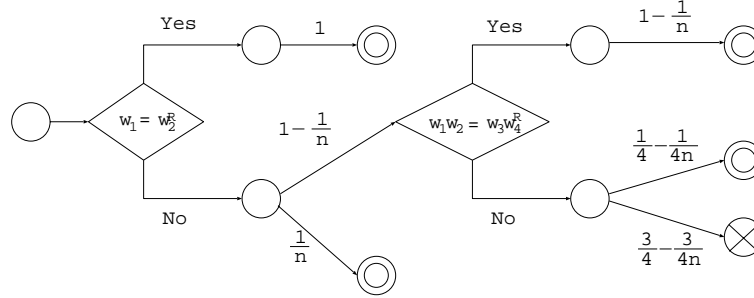


**Fig 4. probability distribution when  $N_1 = N_0\sqrt{n}$ ,  $N_2 = dN'_0$**

then  $M_1^P$  accepts  $x$  with probability at most  $\frac{N_0}{N_1}$  after reading  $w_1 \sharp w_2$  instead of with at most  $(\frac{N_0}{N_1})^2$  in the case of  $M_1^Q$ . There are subtle differences between  $M_1^P$  and  $M_1^Q$ . For example, in the case of  $M_1^P$ , the distributed small amplitudes after reading  $w_3 \sharp w_4$  can be collected completely (there is some loss due to Inverse Fourier transform in the case of  $M_1^Q$ ). This causes a slight difference in the accepting probability of the next lemma (proof is omitted).

**Lemma 8.** Suppose that  $M_1^P$  emulates  $M_1^Q$ . Then  $M_1^P$  accepts strings in  $L_1$  with probability 1 and those not in  $L_1$  with probability at most  $\frac{N_0}{N_1} + \left(1 - \frac{N_0}{N_1}\right) \cdot \frac{N'_0}{N_2}$ . The number of states is approximately the same as the one of  $M_1^Q$ , i.e.,  $\Theta((N_1 N_2)^2 \log N_1 \log N_2)$ . ■

If we set  $N_1 = N_0 n$  and  $N_2 = dN'_0$ , then strings such that  $w_1 \neq w_2^R$  are accepted with probability at most  $\frac{1}{n}$  after reading  $w_1 \sharp w_2$ . Thus this probability is the same as qfa  $M_1^Q$  such that  $N_1 = N_0\sqrt{n}$  and  $N_2 = dN'_0$ , but the lemma says that we need  $\Omega\left(\frac{n^6}{\log^2 n}\right)$  states. See Fig. 5 for a probability distribution.



**Fig 5. probability distribution when  $N_1 = N_0 n$ ,  $N_2 = dN'_0$**

Now we are ready to give our main theorem:

**Theorem 1.** For any integer  $c$ , there is a qfa  $M^Q$  such that  $M^Q$  recognizes  $L(n, n^c)$  and the number of states in  $M^Q$  is  $O\left(\frac{n^{c+4}}{\log^2 n}\right)$ .

**Proof.** The construction of  $M^Q$  is easy: We just add a new deterministic transition from the last accepting state in stage 6 of  $M_1^Q$  to its initial state by reading  $\sharp$ , by which we can manage iteration. Also, we need some small changes to handle the very end of the string: Formally speaking, transition (11) in Table 2 is modified into

$$V_{\sharp}|t_{p_k,0,N_2}\rangle = \frac{1}{\sqrt{N_1}} \sum_{z=1}^{N_1} \exp\left(\frac{2\pi i}{N_2} k z\right) |t_z\rangle,$$

$t_{N_1}$  is now not an accepting state but a non-halting state and two new transitions

$$(10 - c) \quad V_{\S}|q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,e,p_l,f,rej}\rangle$$

$$(12) \quad V_{\sharp}|t_{N_1}\rangle = |q_1\rangle$$

are added.



We set  $N_1 = 2N_0n^{c/2}$  and  $N_2 = dN'_0$ . Then  $N_0/N_1 = \frac{1}{2n^{c/2}}$  and  $\frac{N'_0}{N_2} + \frac{N_0}{N_1} < \frac{1}{2}$  if we select a sufficiently large constant as  $d$ . Suppose that  $M^Q$  has not halted yet and is now reading the  $i$ -th block  $w_{i1}\#w_{i2}\#\#w_{i3}\#w_{i4}$ . Then, we can conclude the following by Lemma 7: (i) If  $w_{i1} = w_{i2}^R$ , then  $M^Q$  accepts the input with probability one. (ii) If  $(w_{i1}w_{i2}) = (w_{i3}w_{i4})^R$ , then (ii - a)  $M^Q$  also accepts the input with probability at most  $1/4n^c$  and (ii - b) rejects the input with at most  $\frac{1}{4n^c} - \frac{1}{16n^{2c}}$  and (ii - c) goes back to the initial state with at least  $1 - \frac{2}{4n^c} + \frac{1}{16n^{2c}}$ . (iii) If  $(w_{i1}w_{i2}) \neq (w_{i3}w_{i4})^R$ , then (iii - a)  $M^Q$  accepts the input with at most  $\frac{1}{4n^c}$ , (iii - b) rejects it with at least  $\frac{3}{4} - \frac{3}{16n^c}$  and (iii - c) goes back to the initial state with at most  $\frac{1}{4} - \frac{1}{16n^c}$ . The number of state is  $O(n^{c+4}/\log^2 n)$ .

Recall that the number of iteration is  $n^c$ . Now suppose that the input  $x$  is in  $L(n, n^c)$ . Then, the probability that  $x$  is rejected is equal to the probability that (ii - b) happens before (i) happens. The probability that (ii - b) happens is at most  $\frac{1}{4n^c}$  per iteration, and so the probability that (ii - b) happens in some iteration is at most  $n^c \cdot \frac{1}{4n^c} = \frac{1}{4}$ . Therefore the probability that  $x$  is finally accepted is well larger than  $1/2$ . Suppose conversely that  $x$  is not in  $L(n, n^c)$ . Then the probability that (ii - a) happens in some iteration is the same as above and is at most  $\frac{1}{4}$ . If  $M^Q$  does not meet a block such that  $(w_{i1}w_{i2}) \neq (w_{i3}w_{i4})^R$  until the end, then the accepting probability is at most this  $1/4$ . If  $M^Q$  does meet such a block in some iteration, it rejects  $x$  with probability at least  $(1 - \frac{1}{4})(\frac{3}{4} - \frac{3}{16n^c})$  which is again well above  $1/2$ . Thus  $M^Q$  recognizes  $L(n, n^c)$ . ■

**Theorem 2.** Suppose that  $M^P$  which emulates  $M^Q$  recognizes  $L_2(n, n^c)$ . Then the number of states of  $M^P$  is  $\Omega(n^{2c+4}/\log^2 n)$ .

**Proof.**  $M^P$  is constructed by applying the same modification (as given in the above proof) to  $M_1^P$ . Then it turns out that we must set  $N'_0/N_2 \leq 1/d$ , where  $d$  is a sufficiently large constant, to reject the strings such that  $(w_{i1}w_{i2}) \neq (w_{i3}w_{i4})^R$  since  $M_1^P$  accepts such bad strings with probability at least  $\frac{N_0}{N_1} + (1 - \frac{N_0}{N_1}) \cdot \frac{N'_0}{N_2}$  by Lemma 8. So we have to set  $N_2 = dN'_0$  and suppose that we set  $N_1 = \frac{1}{a}N_0n^c$ . Then, as shown below,  $M^P$  does not recognize  $L_2(n, n^c)$  if  $a$  is large. That means we have to set  $N_1 = \frac{1}{a}N_0n^c$  for a sufficiently small  $a > 0$ , which implies, from Lemma 8, that we need  $\Omega(n^{2c+4}/\log^2 n)$  states.

Now suppose that the input  $x$  includes a long repetition of blocks such that  $(w_{i1}w_{i2}) = (w_{i3}w_{i4})^R$ . Then  $x$  is accepted in each iteration with probability  $a/n^c$ . Therefore the probability that this happens in the first  $k$  iterations is

$$\sum_{i=1}^k \left(1 - \frac{a}{n^c}\right)^{i-1} \cdot \frac{a}{n^c} = 1 - \left(1 - \frac{a}{n^c}\right)^k.$$

Since the number of repetitions ( $= k$ ) can be as large as  $n^c$ ,

$$\lim_{n \rightarrow \infty} \left(1 - \frac{a}{n^c}\right)^{n^c} = \frac{1}{e^a}.$$

Thus if we select a sufficiently large constant  $a$ , then the probability of acceptance can be arbitrarily close to one. Such an  $M^P$  does not recognize  $L(n, n^c)$  obviously, which proves the theorem. ■

## 5 Concluding Remarks

The question in this paper is whether or not we can exploit the difference in probability calculation between quantum and probabilistic computations. We have shown that the answer is yes using quantum finite automata. However, what remains apparently is whether or not we can exploit this property for other types of models and/or for other types of problems which are preferably less artificial. Also it should be an important future research to obtain a general lower bound for the number of states which is needed to recognize  $L_2(n, n^c)$  by pfa's.

**Acknowledgment.** We are grateful to Mario Szegedy for his many valuable comments to this research. We also thank the anonymous reviewer for Lemma 5. In the earlier version of this paper, the proof of Lemma 7 was very lengthy. Using this lemma and Lemma 6 which is developed under the same idea as Lemma 5, the proof of Lemma 7 was greatly simplified.

$$\begin{aligned}
(1) \quad & V_\ell |q_0\rangle = \frac{1}{\sqrt{N_1 N_2}} \sum_{k=1}^{N_1} \sum_{l=1}^{N_2} |q_{p_k,0,p_l,0,1}\rangle, \\
(2-a) \quad & V_0 |q_{p_k,e,p_l,f,1}\rangle = |q_{p_k,2e,p_l,2f,1}\rangle \quad (0 \leq e < \frac{p_k}{2}, \quad 0 \leq f < \frac{p_l}{2}), \\
(2-b) \quad & V_0 |q_{p_k,e,p_l,f,1}\rangle = |q_{p_k,2e,p_l,2f-p_l,1}\rangle \quad (0 \leq e < \frac{p_k}{2}, \quad \frac{p_l}{2} < f < p_l), \\
(2-c) \quad & V_0 |q_{p_k,e,p_l,f,1}\rangle = |q_{p_k,2e-p_k,p_l,2f,1}\rangle \quad (\frac{p_k}{2} < e < p_k, \quad 0 \leq f < \frac{p_l}{2}), \\
(2-d) \quad & V_0 |q_{p_k,e,p_l,f,1}\rangle = |q_{p_k,2e-p_k,p_l,2f-p_l,1}\rangle \quad (\frac{p_k}{2} < e < p_k, \quad \frac{p_l}{2} < f < p_l), \\
(2-e) \quad & V_1 |q_{p_k,e,p_l,f,1}\rangle = |q_{p_k,2e+1,p_l,2f+1,1}\rangle \quad (0 \leq e < \frac{p_k}{2} - 1, \quad 0 \leq f < \frac{p_l}{2} - 1), \\
(2-f) \quad & V_1 |q_{p_k,e,p_l,f,1}\rangle = |q_{p_k,2e+1,p_l,2f+1-p_l,1}\rangle \quad (0 \leq e < \frac{p_k}{2} - 1, \quad \frac{p_l}{2} - 1 < f < p_l), \\
(2-g) \quad & V_1 |q_{p_k,e,p_l,f,1}\rangle = |q_{p_k,2e+1-p_k,p_l,2f+1,1}\rangle \quad (\frac{p_k}{2} - 1 < e < p_k, \quad 0 \leq f < \frac{p_l}{2} - 1), \\
(2-h) \quad & V_1 |q_{p_k,e,p_l,f,1}\rangle = |q_{p_k,2e+1-p_k,p_l,2f+1-p_l,1}\rangle \\
& \quad (\frac{p_k}{2} - 1 < e < p_k, \quad \frac{p_l}{2} - 1 < f < p_l), \\
(3) \quad & V_\# |q_{p_k,e,p_l,f,1}\rangle = |q_{p_k,e,p_l,f,2}\rangle, \\
(4-a) \quad & V_0 |q_{p_k,e,p_l,f,2}\rangle = |q_{p_k,\frac{e}{2},p_l,2f,2}\rangle \quad (e : \text{even}, \quad 0 \leq f < \frac{p_l}{2}), \\
(4-b) \quad & V_0 |q_{p_k,e,p_l,f,2}\rangle = |q_{p_k,\frac{e}{2},p_l,2f-p_l,2}\rangle \quad (e : \text{even}, \quad \frac{p_l}{2} < f < p_l), \\
(4-c) \quad & V_0 |q_{p_k,e,p_l,f,2}\rangle = |q_{p_k,\frac{e+p_k}{2},p_l,2f,2}\rangle \quad (e : \text{odd}, \quad 0 \leq f < \frac{p_l}{2}), \\
(4-d) \quad & V_0 |q_{p_k,e,p_l,f,2}\rangle = |q_{p_k,\frac{e+p_k}{2},p_l,2f-p_l,2}\rangle \quad (e : \text{odd}, \quad \frac{p_l}{2} < f < p_l), \\
(4-e) \quad & V_1 |q_{p_k,e,p_l,f,2}\rangle = |q_{p_k,\frac{e-1+p_k}{2},p_l,2f+1,2}\rangle \quad (e : \text{even}, \quad 0 \leq f < \frac{p_l}{2} - 1), \\
(4-f) \quad & V_1 |q_{p_k,e,p_l,f,2}\rangle = |q_{p_k,\frac{e-1+p_k}{2},p_l,2f+1-p_l,2}\rangle \quad (e : \text{even}, \quad \frac{p_l}{2} - 1 < f < p_l), \\
(4-g) \quad & V_1 |q_{p_k,e,p_l,f,2}\rangle = |q_{p_k,\frac{e-1}{2},p_l,2f+1,2}\rangle \quad (e : \text{odd}, \quad 0 \leq f < \frac{p_l}{2} - 1), \\
(4-h) \quad & V_1 |q_{p_k,e,p_l,f,2}\rangle = |q_{p_k,\frac{e-1}{2},p_l,2f+1-p_l,2}\rangle \quad (e : \text{odd}, \quad \frac{p_l}{2} - 1 < f < p_l), \\
(5-a) \quad & V_\# |q_{p_k,0,p_l,f,2}\rangle = \frac{1}{\sqrt{N_1}} \sum_{m=1}^{N_1} \exp\left(\frac{2\pi i}{N_1} km\right) |s_{m,0,p_l,f}\rangle, \\
(5-b) \quad & V_\# |q_{p_k,e,p_l,f,2}\rangle = |q_{p_k,e,p_l,f}\rangle \quad (1 \leq e < p_k), \\
(6-a) \quad & V_\# |s_{m,0,p_l,f}\rangle = \frac{1}{\sqrt{N_1}} \sum_{r=1}^{N_1} \exp\left(-\frac{2\pi i}{N_1} mr\right) |q_{p_r,0,p_l,f,3}\rangle \quad (1 \leq m \leq N_1), \\
(6-b) \quad & V_\# |q_{p_k,e,p_l,f}\rangle = |q_{p_k,e,p_l,f,3}\rangle \quad (1 \leq e < p_k), \\
(7-a) \quad & V_0 |q_{p_k,e,p_l,f,3}\rangle = |q_{p_k,2e,p_l,\frac{f}{2},3}\rangle \quad (0 \leq e < \frac{p_k}{2}, \quad f : \text{even}), \\
(7-b) \quad & V_0 |q_{p_k,e,p_l,f,3}\rangle = |q_{p_k,2e,p_l,\frac{f+p_l}{2},3}\rangle \quad (0 \leq e < \frac{p_k}{2}, \quad f : \text{odd}), \\
(7-c) \quad & V_0 |q_{p_k,e,p_l,f,3}\rangle = |q_{p_k,2e-p_k,p_l,\frac{f}{2},3}\rangle \quad (\frac{p_k}{2} < e < p_k, \quad f : \text{even}), \\
(7-d) \quad & V_0 |q_{p_k,e,p_l,f,3}\rangle = |q_{p_k,2e-p_k,p_l,\frac{f+p_l}{2},3}\rangle \quad (\frac{p_k}{2} < e < p_k, \quad f : \text{odd}), \\
(7-e) \quad & V_1 |q_{p_k,e,p_l,f,3}\rangle = |q_{p_k,2e+1,p_l,\frac{f-1+p_l}{2},3}\rangle \quad (0 \leq e < \frac{p_k}{2} - 1, \quad f : \text{even}), \\
(7-f) \quad & V_1 |q_{p_k,e,p_l,f,3}\rangle = |q_{p_k,2e+1,p_l,\frac{f-1}{2},3}\rangle \quad (0 \leq e < \frac{p_k}{2} - 1, \quad f : \text{odd}), \\
(7-g) \quad & V_1 |q_{p_k,e,p_l,f,3}\rangle = |q_{p_k,2e+1-p_k,p_l,\frac{f-1+p_l}{2},3}\rangle \quad (\frac{p_k}{2} - 1 < e < p_k, \quad f : \text{even}), \\
(7-h) \quad & V_1 |q_{p_k,e,p_l,f,3}\rangle = |q_{p_k,2e+1-p_k,p_l,\frac{f-1}{2},3}\rangle \quad (\frac{p_k}{2} - 1 < e < p_k, \quad f : \text{odd}), \\
(8) \quad & V_\# |q_{p_k,e,p_l,f,3}\rangle = |q_{p_k,e,p_l,f,4}\rangle, \\
(9-a) \quad & V_0 |q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,\frac{e}{2},p_l,\frac{f}{2},4}\rangle \quad (e : \text{even}, \quad f : \text{even}), \\
(9-b) \quad & V_0 |q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,\frac{e}{2},p_l,\frac{f+p_l}{2},4}\rangle \quad (e : \text{even}, \quad f : \text{odd}), \\
(9-c) \quad & V_0 |q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,\frac{e+p_k}{2},p_l,\frac{f}{2},4}\rangle \quad (e : \text{odd}, \quad f : \text{even}), \\
(9-d) \quad & V_0 |q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,\frac{e+p_k}{2},p_l,p_l,\frac{f+p_l}{2},4}\rangle \quad (e : \text{odd}, \quad f : \text{odd}), \\
(9-e) \quad & V_1 |q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,\frac{e-1+p_k}{2},p_l,\frac{f-1+p_l}{2},4}\rangle \quad (e : \text{even}, \quad f : \text{even}), \\
(9-f) \quad & V_1 |q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,\frac{e-1+p_k}{2},p_l,\frac{f-1}{2},4}\rangle \quad (e : \text{even}, \quad f : \text{odd}), \\
(9-g) \quad & V_1 |q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,\frac{e-1}{2},p_l,\frac{f-1+p_l}{2},4}\rangle \quad (e : \text{odd}, \quad f : \text{even}), \\
(9-h) \quad & V_1 |q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,\frac{e-1}{2},p_l,\frac{f-1}{2},4}\rangle \quad (e : \text{odd}, \quad f : \text{odd}), \\
(10-a) \quad & V_\# |q_{p_k,0,p_l,0,4}\rangle = \frac{1}{\sqrt{N_2}} \sum_{y=1}^{N_2} \exp\left(\frac{2\pi i}{N_2} ly\right) |t_{p_k,0,y}\rangle, \\
(10-b) \quad & V_\# |q_{p_k,e,p_l,f,4}\rangle = |q_{p_k,e,p_l,f,reg}\rangle \quad (1 \leq f < p_l), \\
(11) \quad & V_\$ |t_{p_k,0,N_2}\rangle = \frac{1}{\sqrt{N_1}} \sum_{z=1}^{N_1} \exp\left(\frac{2\pi i}{N_2} kz\right) |t_z\rangle,
\end{aligned}$$

**Table 2.** state transition diagram of  $M_1^Q$

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